A Remark on Homework 3

As reflected by the grader, the performance of Question 3 from Homework 3 is poor. Let's take a look at its solution and see some common mistakes.

Question. *Suppose that* (x_n) *is a convergent sequence and* (y_n) *is such that for any* $\varepsilon > 0$ *there exists* M *such that* $|x_n - y_n| < \varepsilon$ *for all* $n \geq M$*. Does it follow that* (y_n) *is convergent?*

Solution. Yes. Let L be the limit of (x_n). We shall claim that (y_n) converges to L. Let $\varepsilon > 0$. Since (x_n) converges to L, there exists $N_1 \in \mathbb{N}$ such that

$$|x_n - L| < \frac{\varepsilon}{2}, \quad \forall n \geq N_1. \tag{1}$$

By the assumption, there exists $N_2 \in \mathbb{N}$ such that

$$|x_n - y_n| < \frac{\varepsilon}{2}, \quad \forall n \geq N_2. \tag{2}$$

Take $N = \max\{N_1, N_2\}$. Applying (1), (2) and the triangle inequality, we have

$$|y_n - L| \leq |y_n - x_n| + |x_n - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \forall n \geq N.$$

It follows that (y_n) converges to L by definition.

Remark. Many students get the idea of the proof, but fails to present it clearly. The important "Let $\varepsilon > 0$." cannot be found in their proofs. Notice that in (1) and (2), the choice of N_1 and N_2 both depends on the specified $\varepsilon > 0$. However, if students directly write:

$$\forall \varepsilon > 0, \quad \exists N_1 \in \mathbb{N} \text{ such that } |x_n - L| < \varepsilon, \quad \forall n \geq N_1. \tag{3}$$

$$\forall \varepsilon > 0, \quad \exists N_2 \in \mathbb{N} \text{ such that } |x_n - y_n| < \varepsilon, \quad \forall n \geq N_2. \tag{4}$$

Then (3) and (4) are independent statements. It makes no sense to take $N = \max\{N_1, N_2\}$ because N_1 and N_2 may depends on a different ε. In fact, N_1 and N_2 are not well-defined. On the other hand, some students may write "Let $\varepsilon/2 > 0$." This is not an appropriate way to declare a variable. In the solution, since we specified $\varepsilon > 0$, it follows that $\varepsilon/2 > 0$. We then apply the definition and assumption to the positive number $\varepsilon/2$.

The grader had put a lot of effort to grade your homework assignments and give feed backs. Please make sure you read the notes on comments and common mistakes provided by him every time. "Learning from mistakes" is an effective way to improve the skills on writing proofs. I am sure that it helps the performance on homework and tests if you do so.

Subsequences

Definition (c.f. Definition 3.4.1)**.** Let (x_n) be a sequence of real numbers and let

$$n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$$

be a strictly increasing sequence of **natural numbers**. Then the sequence

$$(x_{n_k}) = (x_{n_1}, x_{n_2}, ..., x_{n_k}, x_{n_{k+1}}, ...)$$

is called a *subsequence* of (x_n).

Remark. The original sequence (x_n) is indexed by n while the subsequence (x_{n_k}) is indexed by k, the n in x_{n_k} indicates that (x_{n_k}) is a subsequence of (x_n).

Example 1. Let (x_n) be a sequence.

- If we take $n_k = 2k$ for each $k \in \mathbb{N}$, we have the subsequence $(x_{n_k}) = (x_2, x_4, x_6, ...)$.
- If we take $n_k = k+10$ for each $k \in \mathbb{N}$, we have the subsequence $(x_{n_k}) = (x_{11}, x_{12}, x_{13}, ...)$.
- If we take $(n_k) = (2, 3, 5, 8, 13, ...)$, we have the subsequence $(x_{n_k}) = (x_2, x_3, x_5, x_8, x_{13}, ...)$.

Theorem (c.f. Theorem 3.4.2)**.** *Let (x_n) be a sequence of real numbers that converges to $x \in \mathbb{R}$. Then every subsequence (x_{n_k}) of (x_n) also converges to x.*

Example 2 (c.f. Section 3.4, Ex.7)**.** Establish the convergence and find the limits of the following sequences:

(a) $((1 + 1/2n)^n)$ (b) $\left((1 + 1/n^2)^{2n^2}\right)$

Solution. Recall the definition of the **Euler number**: (see **Example 3.3.6**)

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

(a) Note that if we take $m = 2n$,

$$\left(1 + \frac{1}{2n}\right)^n = \left(1 + \frac{1}{m}\right)^{m/2} = \sqrt{\left(1 + \frac{1}{m}\right)^m}.$$

Hence each term of the given sequence is the square-root of each term of a subsequence of $((1 + 1/n)^n)$. Hence the given sequence converges to $\sqrt{e}$.

(b) Note that if we take $m = n^2$,

$$\left(1 + \frac{1}{n^2}\right)^{2n^2} = \left(1 + \frac{1}{m}\right)^{2m} = \left[\left(1 + \frac{1}{m}\right)^m\right]^2.$$

Hence each term of the given sequence is the square of each term of a subsequence of $((1 + 1/n)^n)$. Hence the given sequence converges to e^2.

Theorem (c.f. Theorem 3.4.4)**.** *Let (x_n) be a sequence of real numbers and $x \in \mathbb{R}$. Then the following statements are equivalent:*

(i) The sequence (x_n) does not converges to x.

(ii) There exists an $\varepsilon_0 > 0$ such that for any $k \in \mathbb{N}$, there exists natural number $n_k > k$ such that $|x_{n_k} - x| \geq \varepsilon_0$.

(iii) There exists an $\varepsilon_0 > 0$ and a subsequence (x_{n_k}) of (x_n) such that $|x_{n_k} - x| \geq \varepsilon_0 \quad \forall k$.

Divergence Criteria (c.f. 3.4.5)**.** *If a sequence (x_n) of real numbers has either of the following properties, then (x_n) is divergent.*

(i) (x_n) is unbounded.

(ii) (x_n) has a divergent subsequence.

(iii) (x_n) has two convergent subsequences (x_{n_k}) and (x_{r_k}) whose limits are not equal.

Example 3 (c.f. Examples 3.4.6)**.** The sequence $(x_n) = ((-1)^n)$ is divergent. If we take $n_k = 2k$ and $r_k = 2k-1$, then subsequences (x_{n_k}) and (x_{r_k}) satisfies:

$$(x_{n_k}) = (x_{2k}) = (1, 1, 1, ...) \quad \text{and} \quad (x_{r_k}) = (x_{2k-1}) = (-1, -1, -1, ...)$$

Their limits are 1 and -1 respectively, hence (x_n) is divergent.

Example 4 (c.f. Section 3.4, Ex.9)**.** Suppose that every subsequence of (x_n) has a subsequence that converges to 0. Show that $\lim(x_n) = 0$.

Solution. Suppose on a contrary that $\lim(x_n) \neq 0$. Hence by **Theorem 3.4.4**, there exists $\varepsilon_0 > 0$ and a subsequence (x_{n_k}) of (x_n) such that

$$|x_{n_k} - 0| = |x_{n_k}| \geq \varepsilon_0, \quad \forall k \in \mathbb{N}. \tag{5}$$

By assumption, there is a subsequence $(x_{n_{k_j}})$ of (x_{n_k}) such that $\lim_j(x_{n_{k_j}}) = 0$. i.e., there exists $J \in \mathbb{N}$ such that

$$|x_{n_{k_j}} - 0| = |x_{n_{k_j}}| < \varepsilon_0 \quad \forall j \geq J.$$

In particular, take $K = k_J$. Then $|x_{n_K}| < \varepsilon_0$ and it contradicts (5).

Example 5 (c.f. Section 3.4, Ex.14)**.** Let (x_n) be a bounded sequence and let $s = \sup(x_n)$. Show that if $s \notin \{x_n : n \in \mathbb{N}\}$, then there is a subsequence of (x_n) that converges to s.

Solution. We need to pick the subsequence (x_{n_k}) by choosing a strictly increasing sequence of natural numbers $n_1 < n_2 < \cdots$. By definition of supremum, take $n_1 \in \mathbb{N}$ such that

$$x_{n_1} > s - 1.$$

Then we claim that there exists an integer $n_2 > n_1$ such that

$$x_{n_2} > s - \frac{1}{2}.$$

Suppose not. Then $x_n \leq s - 1/2$ for any $n > n_1$. It forces $x_n = s$ for some $n \leq n_1$, but it contradicts the assumption that $s \notin \{x_n : n \in \mathbb{N}\}$. Similarly, for $k = 2, 3, ...$, we can pick $n_{k+1} > n_k$ such that

$$x_{n_{k+1}} > s - \frac{1}{k+1}.$$

It follows by **Squeeze Theorem** that $\lim(x_{n_k}) = s$ because

$$s - \frac{1}{k} < x_{n_k} \leq s, \quad \forall k \in \mathbb{N}.$$

Example 6 (c.f. Section 3.4, Ex.12)**.** Show that if (x_n) is unbounded, then there exists a subsequence (x_{n_k}) such that $\lim(1/x_{n_k}) = 0$.

Solution. We aim to construct a subsequene (x_{n_k}) such that

$$0 \le \left|\frac{1}{x_{n_k}}\right| < \frac{1}{k}, \quad \forall k \in \mathbb{N}\,. \tag{6}$$

If we can do so, then $\lim(1/x_{n_k}) = 0$ by the **Squeeze Theorem**. Notice that (6) is equivalent to $|x_{n_k}| > k$, which suggest to make use of the condition that the sequence (x_n) is unbounded.

For $k = 1$, since (x_n) is unbounded, there exists $n_1 \in \mathbb{N}$ such that $|x_{n_1}| > 1$. For $k = 2$, since (x_n) is unbounded, there exists $n_2 > n_1$ such that $|x_{n_2}| > 2$. Notice that we can choose $n_2 > n_1$ because otherwise we have

$$|x_n| \le 2, \quad \forall n > n_1.$$

This implies that (x_n) is bounded by $\max\{|x_1|, |x_2|, ..., |x_{n_1}|, 2\}$, contradicts the fact that (x_n) is unbounded. Repeat the same argument, we can find a strictly increasing sequence of natural numbers $n_1 < n_2 < \cdots$ such that

$$|x_{n_k}| > k \implies \left|\frac{1}{x_{n_l}}\right| < \frac{1}{k}, \quad \forall k \in \mathbb{N}\,.$$

The result follows.